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On the Epicycloid.

By F. Morley.

The detailed properties of the hypocycloid of class 3 have received much attention since Cremona's paper (Crelle, Vol. 64). See Nouvelles Annales, Series 2, Vol. IX, for papers by Serret, Painvin, Laguerre; Quarterly Journal, IX; Giornale di Mat., XXIII; Messenger, XII (MacMahon); American Journal, X, 3 (Humbert) The peculiar behavior of the general algebraic hypocycloid or epicycloid at infinity being easily determined, few recent mathematicians have stooped to pick up the special properties of these curves. It seems worth while to prove some of these details, using the method of circular coordinates, here very convenient. Some of the results are Wolstenholme's (Proc. London Math. Soc., Vol. 4); most are in line with the very general properties of algebraic curves recently proved by Humbert (referred to later), though the method of proof used in the very simple cases here considered is quite different.

Let, in the epicycloid,

$a \equiv$ radius of fixed or cusp-circle,

$b \equiv$ radius of moving circle,

$c \equiv a + 2b \equiv$ radius of vertex circle,

$\phi \equiv$ inclination of the radius to the point of contact of the circles (a) and (b), the zero line being the radius to a vertex,

$p/q \equiv (c+a)/(c-a)$, p and q being integers prime to each other unless $q = 1$, and p being $> q$.

We have, as in Salmon, Higher Plane Curves, third edition, §305,

$$X = mb \cos\phi + b \cos m\phi,$$
$$Y = mb \sin\phi + b \sin m\phi,$$

where $m \equiv p/q$. Hence writing

$$x \equiv X + iY,$$
$$y \equiv X - iY,$$
$$z \equiv e^{\phi i} \equiv \zeta^q,$$

and noting that $b \equiv (c-a)/2 \equiv qc/(p+q)$, we have

$$\left.\begin{aligned}(p+q)\,x/c &= p\zeta^q + q\zeta^p \\ (p+q)\,y/c &= p\zeta^{-q} + q\zeta^{-p}\end{aligned}\right\}. \qquad (1)$$

Hence the degree of the curve is $2p$.

The tangent is

$$\begin{vmatrix} x & y & 1 \\ p\zeta^{q}+q\zeta^{p} & p\zeta^{-q}+q\zeta^{-p} & (p+q)/c \\ pq\,(\zeta^{q-1}+\zeta^{p-1}) & -pq\,(\zeta^{-q-1}+\zeta^{-p-1}) & 0 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} x & y\zeta^{p+q} & 1 \\ p\zeta^{q}+q\zeta^{p} & p\zeta^{p}+q\zeta^{q} & (p+q)/c \\ 1 & -1 & 0 \end{vmatrix} = 0,$$

or

$$x+y\zeta^{p+q}=c\,(\zeta^{p}+\zeta^{q}). \tag{2}$$

Hence the class of the curve is $p+q$.

For the hypocycloid we have, changing the sign of q since now $c<a$,

$$\left.\begin{aligned} (p-q)\,x/c &= p\zeta^{-q}-q\zeta^{p} \\ (p-q)\,y/c &= p\zeta^{q}\ \ -q\zeta^{-p} \end{aligned}\right\}, \tag{3}$$

and

$$x\zeta^{q}+y\zeta^{p}=c\,(1+\zeta^{p+q}). \tag{4}$$

Hence the degree and class are still $2p$ and $p+q$.

The forms of the curves at infinity are readily got from the above equations, and we see that the only points at infinity are the points I, J, each being counted p times. In the epicycloid the singular tangents at I, J are directed to the origin, where all the foci are collected. The centre being the mean of the foci, the origin is also the centre of the curve. In the hypocycloid the line IJ is the singular tangent at both I and J, and there are no finite foci. See Wolstenholme, Proc. London Math. Soc., IV, p. 327; S. Roberts, ib., pp. 354, 355.

Thus in the hypocycloid of class 4, $p+q=4$, $\therefore p=3$, $q=1$; there are cusps at I, J, and IJ is the common cusp-tangent. One sees that injustice is done to the points IJ by calling this curve (as college text-books are apt to do) the four-cusped hypocycloid.

We may note that for $p+q<5$ or $=6$ there is only one epi- or hypocycloid of given class. I shall distinguish the case $q=1$ (when, since $p/q=(c+a)/(c\sim a)$ $=(a\pm b)/b$, b is a divisor of a and the curve is described in one revolution of moving circle) from the other cases by calling the latter nodal, since there are nodes when $q>1$.

If we draw all the tangents from a point x, y we have from (2)

$$\begin{aligned} \text{Product of } \zeta\text{'s} &= (-)^{p+q}x/y \\ &= (-)^{p+q}e^{2\theta i}, \end{aligned}$$

where θ is the inclination of the radius to x, y. And if θ_r is the inclination of ζ_r,

$$\zeta_r^{p+q}=-e^{2\theta_r i},\quad r=1 \text{ to } p+q.$$

Hence

$$(-)^{p+q}e^{2\Sigma\theta_r i}=(-)^{(p+q)^2}e^{2(p+q)\theta i}$$

and

$$\Sigma\theta_r=(p+q)\,\theta.$$

But in the case of the hypocycloid we have in the same way from (4),

$$\Sigma\theta_r = 0.$$

These are cases of Laguerre's theorem (Humbert, American Journal, X, p. 262). They are given by F. R. J. Hervey (Educational Times Reprint, L, p. 110). His method of proof is virtually that used here.

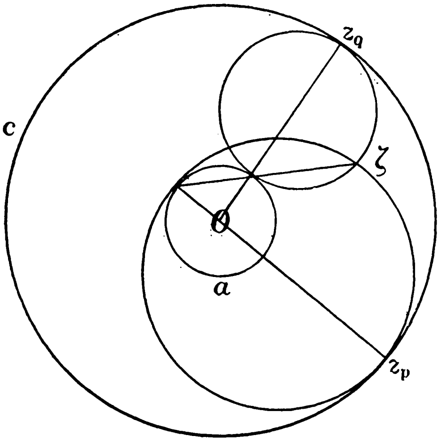
Where (2) meets the vertex circle, $xy = c^2$, so that

$$x^2 + c^2\zeta^{p+q} = cx(\zeta^p + \zeta^q)$$

and

$$\left.\begin{matrix} x = c\zeta^p \text{ or } c\zeta^q \\ y = c\zeta^{-p} \text{ or } c\zeta^{-q} \end{matrix}\right\}. \qquad (5)$$

These are the points of contact with the vertex circle of the two generating circles through the point ζ (see fig.).



I shall call the generation by the smaller circle of radius $(c-a)/2$ direct, and that by the larger circle of radius $(c+a)/2$ indirect. A point on the vertex circle is defined by z or $e^{\phi i}$; let the direct point z corresponding to ζ be z_q, the indirect z_p. Then from the equations preceding (1),

$$z_q = \zeta^q, \text{ and similarly } z_p = \zeta^p.$$

Hence there are q direct and p indirect tangents from a point on the vertex circle. For the points of contact of the former, (1) gives

$$(p+q)x/c = pz_q + q\zeta^p,$$

so that they are the vertices of a regular polygon in a circle whose centre is on the radius Oz_q at a distance $pc/(p+q) \equiv (c+a)/2$ from O, and whose radius is $qc/(p+q)$ or $(c-a)/2$; and this is the direct circle which touches the vertex circle at z_q. Hence we have a theorem of Wolstenholme's that *the q direct points* *of contact of tangents from a point* z *on the vertex circle form a regular polygon in a direct circle; the same holds for the* p *indirect tangents; and the two circles touch the vertex circle at* z (Wolstenholme, ut supra, p. 322). It is evident that the tangents form two equiangular pencils. This is a familiar fact for the hypocycloid of class 3, in the form that perpendicular tangents meet on the inscribed circle. It is given for the class 4 by R. A. Roberts, Problems on Curves, p. 180.

Start with a point z and draw the direct tangents. They meet the vertex circle again at points z_p such that $z_p = \zeta^p$, $\zeta^q = z$, $z_p = z^{p/q}$.

If we again draw the direct tangents from the points z_p so given, we get new points $z_{pp} = z_p^{p/q} = z^{p^2 q^2}$, etc. Thus the direct tangents from a point on the vertex circle meet the circle again at the vertices of a regular q-gon; the direct tangents from these points meet the circle in a regular q^2-gon and so on. The first q tangents of course form an equiangular system of lines. But any succeeding set has the same property. For $e^{2\theta i} = -\zeta^{p+q}$ from (2). And

$$\zeta^q = z_p = z^{p/q}.$$

Hence
$$e^{2\theta i} = -z^{p(p+q)/q^2}$$

Hence the angle between two tangents of the second set is $r\pi/q^2$.

This is clearly true for any set of direct or indirect tangents.

If we start with a regular n-gon in the circle, where to avoid coincidence of points n is supposed prime to p and q, we have, if the polygon is given by $z^n = z_0$,

$$z_p = z^{p/q} = z_0^{p/qn},$$

giving a regular qn-gon,

$$z_{pp} = z_p^{p/q} = z_0^{p^2/q^2 n},$$

giving a regular $q^2 n$-gon, and so on.

We have as before for the inclination of any tangent of the first set

$$e^{2\theta i} = -z_0^{(p+q)/nq}.$$

Hence the first set is equiangular, and so in general except when $p+q$ is a multiple of n.

Let, now, $n = p+q$. Then the last equation gives only q values of θ.

Hence, *if from the vertices of a regular* $(p+q)$*-gon in the circle we draw the direct tangents, they form* q *parallel sets.* And from (2) for parallel tangents ζ^{p+q} is given, so that *parallel tangents meet the circle in two regular* $(p+q)$*-gons.*

It is evident geometrically from Wolstenholme's theorem above (see fig.), or comes at once from equations (1), that each tangent, considered as a chord of the vertex circle, is divided in the ratio q/p at the point of contact.

Hence the points of contact of parallel tangents are the orthogonal projections of a regular polygon.

One immediate inference is the truth for the epicycloid of Chasles' theorem that the mean of the points of contact of parallel tangents is fixed. Others are that the area obtained by joining (in any definite order) the points of contact is constant; and that *the points of contact of parallel tangents lie on an ellipse whose auxiliary circles are the vertex circle and cusp circle, and whose minor axis is parallel to the tangents.*

When tangents are drawn from any point x, y, we have from (2), if $z = \zeta^q$,

$$x + yz^{1+p/q} = c\,(z + z^{p/q}),$$

$$(x - cz)^q = (c - yz)^q z^p,$$

whence, if $p > 1$, $$\Sigma\zeta^q = \Sigma z = qc/y.$$

Similarly, if $q > 1$, $$\Sigma\zeta^p = pc/y.$$

Hence, substituting in (1), the mean centre of the points of contact is given by

$$(p + q)^2\,\bar{x}/c = 2pqc/y,$$

and similarly $(p + q)^2 \bar{y}/c = 2pqc/x$. Hence *the mean centre of the points of contact of tangents from any point P to a nodal epicycloid is the inverse of P with regard to a fixed circle.*

The radius of the circle is

$$\frac{\sqrt{2pq}}{p + q}\,c, \text{ or } \sqrt{(c^2 - a^2)/2}.$$

A similar calculation for the hypocycloid, starting with (4), gives, if $p > 1$, $q > 1$,

$$\Sigma\zeta^p = px/c,\ \Sigma\zeta^q = qy/c,\ \Sigma\zeta^{-p} = py/c,\ \Sigma\zeta^{-q} = qx/c,$$

and hence from (3), $\bar{x} = \bar{y} = 0$, or *the mean centre of the points of contact from any point to a nodal hypocycloid is fixed.*

This fact is interesting as showing that the enunciation of a theorem of Humbert's (Liouville, 4th series, Vol. III, p. 363) may be generalized. There, among a crowd of important theorems, he states the above fact for any curve which the line infinity meets at points of inflexions but does not touch; but the hypocycloid is not of such a kind.

In the epicycloid the perpendicular from the origin on the tangent is

$$w = 1/2 . c . (\zeta^{(p-q)/2} + \zeta^{(q-p)/2}).$$

Now, for all the tangents from a given point we have from (1),

$$\Pi\,(1 - \zeta_r/\zeta) \equiv 1 - c\,(\zeta^{-p} + \zeta^{-q})/y + x\zeta^{-p-q}/y,$$

$$r = 1 \text{ to } p + q.$$

Calculating $\Sigma\zeta_r^{p-q}$ by taking logarithms, we see that when $q > 1$, since $p-q$ is prime to p and q, the sum is zero. So for $\Sigma\zeta_r^{q-p}$. Hence

$$\Sigma w^2 = (p+q)\,c^2/2. \tag{6}$$

Further, if $p-q$ is even, $\Sigma\zeta_r^{(p-q)/2} = 0$, and $\Sigma w = 0$. Hence, *in nodal epicycloids of even class the sum of the distances from the centre on the tangents from any point is zero.*

Since the radius of curvature is proportional to w, such theorems may be also stated for radii of curvature. And since the normal envelops a similar concentric curve, we have, if w' be the perpendicular from the centre on a normal,

$$\Sigma w'^2 = \text{const.},$$

and hence *in a nodal epicycloid the sum of the squares of the distances from the centre to the points of contact of tangents from any point is constant.*

The same statements hold for hypocycloids.

Two tangents cut at a constant angle if the ratio of their parameters ζ is given. Let them be

$$x + y\zeta^{p+q} = c\,(\zeta^p + \zeta^q),$$
$$k^{p+q}x + y\zeta^{p+q} = c\,(k^q\zeta^p + k^p\zeta^q),$$
$$\therefore\ x\,(1-k^{p+q}) = c\,(1-k^q)\,\zeta^p + c\,(1-k^p)\,\zeta^q,$$
$$y\,(1-k^{p+q}) = ck^q\,(1-k^p)\,\zeta^{-q} + ck^p\,(1-k^q)\,\zeta^{-p},$$

or

$$x = A\zeta_1^p + B\zeta_1^q,$$
$$y = A\zeta_1^{-p} + B\zeta_1^{-q},$$

where

$$\zeta_1 = \zeta/k^{p/2},$$
$$A = ck^{p/2}\,(1-k^q)/(1-k^{p+q}),$$
$$B = ck^{q/2}\,(1-k^p)(1-k^{p+q}).$$

These are the equations of an epitrochoid in which

$-A$ = distance of generating point from moving centre,

B = sum of radii of fixed and moving circles (Salmon, Curves, §305),

and the ratio in our case of the radii of the fixed and moving circles = that for the epicycloid.

The result is Wolstenholme's (p. 330).

As Wolstenholme indicates (p. 418, see also p. 327), it is not complete when the constant angle is $r\pi/p$ or $r\pi/q$. The vertex circle is then part of the locus. The above equations show this by the vanishing of $1-k^p$ or $1-k^q$. For let the angle be $r\pi/p$. Then

$$k^{p+q} = e^{2r\pi i/p},$$
$$k^{p(p+q)} = 1,$$

and one value of k^p is 1, giving the vertex circle. The other values give an epitrochoid.

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